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Momentum and angular momentum in predictive relativistic electrodynamics

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Abstract. The formulae for the total momentum and angular momentum of a system of two point-charges in Wheeler–Feynman electrodynamics are integrated using the approximate parametric equations of the trajectories in predictive relativistic mechanics. The zero- and first-order terms in the expansion in power series of $g \equiv e_1e_2$ are evaluated exactly, and the second-order term is calculated approximately for slow motion.

1. Introduction

There are well known formulae (Wheeler and Feynman 1949, Dettman and Schild 1954, Anderson 1967) for the determination of the momentum and angular momentum of a system of point-like charged particles in Wheeler–Feynman electrodynamics. However, these formulae involve integrations along the trajectories of the particles, and therefore they cannot be applied without prior knowledge of these trajectories. Since the basic assumption of predictive relativistic mechanics (PRM) is that the trajectories of the particles are completely determined by their initial positions and velocities, it is possible, in PRM, to give formulae for the momentum and angular momentum in terms of only these initial data. This is precisely the object of this paper, for the two-particle case, though it has been carried out only in an approximate way.

First of all, we need the equations of the trajectories. Bel and Martin (1973a) have given a recurrent method to obtain these trajectories up to any order in some coupling constant g, provided the four-acceleration functions of the covariant formalism are known up to the same order (Bel *et al* 1973), without having to integrate the second-order system of differential equations of motion. In § 2 we modify their method somewhat, choosing a parameter more suited to our needs.

In the following sections we give general expressions for the four-momentum and angular momentum, and we evaluate them up to order g exactly and the terms of order g^2 approximately (for slow motion). In the conclusion the results obtained are given more explicitly, for easier reference.

Evidently, the method employed to perform integrations along the trajectories, using the parameter defined in § 2, is also applicable to many other problems in PRM.

2. Parametric equations of the trajectories

We consider a system of two point-like particles whose trajectories are given by the following system of eight second-order differential equations in the Minkowskian space–time M_4†:

$$\ddot{z}_a^\alpha = \xi_a^\alpha(z_c^\beta, \dot{z}_d^\gamma), \tag{2.1}$$

whose general solution is

$$z_a^\alpha = \psi_a^\alpha(x_c^\beta, u_d^\gamma; \tau)$$

with

$$\psi_a^\alpha(x_c^\beta, u_d^\gamma; 0) = x_a^\alpha, \qquad \dot{\psi}_a^\alpha(x_c^\beta, u_d^\gamma; 0) = u_a^\alpha.$$

If τ is to be the proper time along each trajectory, the u_a^α's are the initial four-velocities and they are therefore subjected to the following restrictions:

$$u_a^0 > 0, \qquad u_a^\rho u_{a\rho} = -1.$$

Let us suppose: (i) that the two-particle system is isolated, which implies that system (2.1) is Poincaré-invariant; and (ii) that system (2.1) is a predictive system (Bel *et al* 1973), that is, that the initial conditions x_a^α, u_b^β yield the same trajectories as the initial conditions $\bar{x}_a^\alpha \equiv \psi_a^\alpha(x_c^\beta, u_d^\gamma; \tau_a)$, $\bar{u}_a^\alpha \equiv \dot{\psi}_a^\alpha(x_c^\beta, u_d^\gamma; \tau_a)$ for any τ_1, τ_2. The Poincaré invariance is equivalent to

$$\xi_a^\alpha[L_\lambda^\beta(z_c^\lambda - A^\lambda), L_\mu^\gamma \dot{z}_d^\mu] = L_\mu^\alpha \xi_a^\mu(z_c^\beta, \dot{z}_d^\gamma), \tag{2.2}$$

and the predictivity condition is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau_b}\xi_a^\alpha \equiv \dot{z}_b^\rho \frac{\partial \xi_a^\alpha}{\partial z_b^\rho} + \xi_b^\rho \frac{\partial \xi_a^\alpha}{\partial \dot{z}_b^\rho} = 0 \tag{2.3}$$

and

$$\dot{z}_a^\rho \xi_{a\rho} = 0. \tag{2.4}$$

The functions ξ^α that satisfy Poincaré invariance are vectors that may be written in the form (Arens 1972)

$$\xi_a^\alpha = a_a x_{ab}^\alpha + b_{aa} u_a^\alpha + b_{ab} u_b^\alpha, \tag{2.5}$$

where a_a and b_{ac} are functions of the four independent scalars

$$k \equiv -(u_1 u_2), \qquad r_b \equiv [x_{ab}^2 + (x_{ab}u_b)^2]^{1/2},$$
$$s_b \equiv (x_{ab}u_a) + (u_1 u_2)(x_{ab}u_b); \qquad y_b \equiv (x_{ab}u_b). \tag{2.6}$$

We shall suppose that these functions are known as expansions in power series of some

† $a, b, c, \ldots = 1, 2$; a and b will always be different, and there will never be summation over these indices, but $c, d, \ldots$ can take the same values as a or b, and they follow the summation convention.

$\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$; $i, j, k, \ldots = 1, 2, 3$. We use the signature $+2$ for M_4, and unless otherwise stated, the units are chosen so that $c = 1$.

z_c^α and $\dot{z}_d^\beta$ are the positions and velocities along the trajectories, and x_c^α, u_d^β are reserved for the initial positions and velocities.

$$z_{ab}^\alpha \equiv z_a^\alpha - z_b^\alpha, \qquad x_{ab}^\alpha \equiv x_a^\alpha - x_b^\alpha.$$

coupling constant g (the first terms of these expansions have been computed for the electromagnetic interaction (Bel *et al* 1973, Salas and Sanchez Ron 1974) and for the short-range scalar interaction (Bel and Martin 1974), taking in both cases $g = e_1 e_2$; they have also been computed for the electromagnetic interaction taking $g = m_1/m_2$ (Bel and Martin 1973b), but this is not properly a coupling constant, since there is interaction even when $g = 0$).

What we are now interested in is to find the parametric equations of the trajectories without integrating system (2.1). Bel and Martin (1973a) have given a method to obtain these equations, but it is only locally valid because the parameter they use does not vary monotonically along the trajectory (however their final results are globally correct for non-positive values of their parameter C^2; for $C^2 = 0$ they have been successfully used by Salas and Sanchez Ron (1974)). In what follows we are going to apply their method, but choosing a parameter θ for each trajectory which is a monotonically increasing function of the proper time. This parameter is defined by

$$u_{a\rho}[z_b^\rho(\theta_b) - x_a^\rho] \equiv -\theta_b. \tag{2.7}$$

Let us write

$$z_a^\alpha - x_b^\alpha = \rho_b x_{ba}^\alpha + \sigma_{ba} u_a^\alpha + \sigma_{bb} u_b^\alpha, \tag{2.8}$$

where z_b^α, ρ_b and σ_{bc} are functions of the initial conditions x_c^β, u_d^γ and the parameter θ_b. It is clear that $(\mathrm{d}/\mathrm{d}\tau_b) z_b^\alpha = 0$, and from this equation and the following expansions†:

$$\xi_a^\alpha = \sum_{n=1}^{\infty} g^n \xi_a^{(n)\alpha}, \qquad a_a = \sum_{n=1}^{\infty} g^n a_a^{(n)}, \qquad b_{ac} = \sum_{n=1}^{\infty} g^n b_{ac}^{(n)}$$
$$\rho_a = \sum_{n=0}^{\infty} g^n \rho_a^{(n)}, \qquad \sigma_{ac} = \sum_{n=0}^{\infty} g^n \sigma_{ac}^{(n)}, \tag{2.9}$$

there follow (Bel and Martin 1973a)

$$\left.\begin{aligned} \frac{\partial \rho_b^{(n)}}{\partial y_b} &= -\sum_{p+q=n} \left(a_b^{(p)} \sigma_{bb}^{(q)} + \xi_b^{(p)\rho} \frac{\partial \rho_b^{(q)}}{\partial u_b^\rho} \right) \\ \frac{\partial \sigma_{bc}^{(n)}}{\partial y_b} &= -\sum_{p+q=n} \left(b_{bc}^{(p)} \sigma_{bb}^{(q)} + \xi_b^{(p)\rho} \frac{\partial \sigma_{bc}^{(q)}}{\partial u_b^\rho} \right) - \delta_{bc} \rho_b^{(n)} \end{aligned}\right\} \quad (n > 0) \tag{2.10}$$

and

$$\frac{\partial \rho_b^{(0)}}{\partial y_b} = 0, \qquad \frac{\partial \sigma_{bc}^{(0)}}{\partial y_b} = -\delta_{bc}(1 + \rho_b^{(0)}). \tag{2.11}$$

On the other hand, from (2.7) and (2.8), we have

$$\sigma_{bb} = k^{-1}(y_a \rho_b - \sigma_{ba} + y_a + \theta_b). \tag{2.12}$$

The equation (2.10) with left-hand term $\partial \sigma_{bb}/\partial y_b$ is a consequence of the other two equations (2.10) and equation (2.11) and, therefore, it can be omitted.

† $f^{(n)}$ will always denote the coefficient of g^n in the expansion of the function f into power series of g, so that $f = \sum_{n=0}^{\infty} g^n f^{(n)}$.

For $\tau_b = 0$, we have $z_b^\alpha = x_b^\alpha$, and therefore $\theta_b = s_b + ky_b$, and $\rho_b = 0, \sigma_{bc} = 0$. Hence, integrating (2.10) with these initial conditions for $y_b = k^{-1}(\theta_b - s_b)$, we finally obtain

$$\rho_b^{(n)} = -\int_{k^{-1}(\theta_b - s_b)}^{y_b} \sum_{p+q=n} \left(a_b^{(p)}\sigma_{bb}^{(q)} + \xi_b^{(p)\rho}\frac{\partial \rho_b^{(q)}}{\partial u_b^\rho} \right) \mathrm{d}y_b \qquad (n > 0)$$

$$\sigma_{bc}^{(n)} = -\int_{k^{-1}(\theta_b - s_b)}^{y_b} \left[\sum_{p+q=n} \left(b_{bc}^{(p)}\sigma_{bb}^{(q)} + \xi_b^{(p)\rho}\frac{\partial \sigma_{bc}^{(q)}}{\partial u_b^\rho} \right) + \delta_{bc}\rho_b^{(n)} \right] \mathrm{d}y_b \tag{2.13}$$

where the integrands must be expressed in terms of k, r_b, s_b and y_b, and the first three scalars remain constant during the integration, as well as the parameter θ_b.

For $n = 0$ it is immediately found

$$\rho_b^{(0)} = 0, \qquad \sigma_{ba}^{(0)} = 0, \qquad \sigma_{bb}^{(0)} = k^{-1}(y_a + \theta_b). \tag{2.14}$$

In the case of the electromagnetic interaction with $g = e_1 e_2$ (Lienard–Wiechert or Wheeler–Feynman), remembering that $a_b^{(1)} = m_b^{-1}kr_a^{-3}$, $b_{ba}^{(1)} = -m_b^{-1}r_a^{-3}y_b$, $b_{bb}^{(1)} = 0$, (Bel *et al* 1973, Salas and Sanchez Ron 1974) there result for $n = 1$

$$\rho_b^{(1)} = m_b^{-1}\int_{k^{-1}(\theta_b - s_b)}^{y_b} r_a^{-3}(ky_b + s_b - \theta_b)\,\mathrm{d}y_b$$

$$\sigma_{ba}^{(1)} = m_b^{-1}k^{-1}\int_{k^{-1}(\theta_b - s_b)}^{y_b} r_a^{-3}y_b(-ky_b - s_b + \theta_b)\,\mathrm{d}y_b, \tag{2.15}$$

with

$$r_a = [r_b^2 + s_b^2 + 2ks_b y_b + (k^2-1)y_b^2]^{1/2}.$$

The parameter θ that we have introduced would be adequate for the following calculations, but it is preferable to employ a new parameter which not only is a monotonically increasing function of the proper time, but also takes zero value when $\tau = 0$, that is, when $z^\alpha = x^\alpha$. To this end, we define

$$\lambda_b \equiv k^{-1}(y_a + \theta_b), \tag{2.16}$$

and the new parameter λ has the supplementary advantage of coinciding with the proper time to order zero.

With the new parameter we have

$$\sigma_{bb} = k^{-1}(y_a\rho_b - \sigma_{ba}) + \lambda_b \tag{2.17}$$

$$\rho_b^{(0)} = 0, \qquad \sigma_{ba}^{(0)} = 0, \qquad \sigma_{bb}^{(0)} = \lambda_b. \tag{2.18}$$

For the formulae corresponding to $n = 1$ for the electromagnetic interaction with $g = e_1 e_2$, we first define

$$J_b^r(\lambda_b) \equiv \int_{y_b}^{y_b + \lambda_b} [r_b^2 + s_b^2 + 2ks_b y_v + (k^2 - 1)y^2]^{-3/2} y^r\,\mathrm{d}y, \tag{2.19}$$

and with this convention the formulae are

$$\rho_b^{(1)} = m_b^{-1}k[-J_b^1 + (y_b + \lambda_b)J_b^0]$$

$$\sigma_{ba}^{(1)} = m_b^{-1}[J_b^2 - (y_b + \lambda_b)J_b^1] \tag{2.20}$$

$$\sigma_{bb}^{(1)} = m_b^{-1}k^{-1}[-J_b^2 + (ks_b + (k^2+1)y_b + \lambda_b)J_b^1 + ky_a(y_b + \lambda_b)J_b^0].$$

3. Four-momentum and angular momentum in Wheeler–Feynman electrodynamics

In Wheeler–Feynman electrodynamics, the four-momentum and the angular momentum about the origin of a system of two point charges are (Wheeler and Feynman 1949, Dettman and Schild 1954, Anderson 1967)

$$p^\mu = \sum_a [m_a u_a^\mu + e_a A_b^\mu(x_a^\lambda)] + 2e_1e_2\left(\int_0^{+\infty} \mathrm{d}\lambda_1 \int_{-\infty}^0 \mathrm{d}\lambda_2 - \int_{-\infty}^0 \mathrm{d}\lambda_1 \int_0^{+\infty} \mathrm{d}\lambda_2\right)\delta'(z_{12}^2)z_{12}^\mu(z_1'z_2'), \tag{3.1}$$

$$L^{\mu\nu} = \sum_a \{x_a^\mu[m_a u_a^\nu + e_a A_b^\nu(x_a^\lambda)] - x_a^\nu[m_a u_a^\mu + e_a A_b^\mu(x_a^\lambda)]\} + 2e_1e_2\left(\int_0^{+\infty} \mathrm{d}\lambda_1 \int_{-\infty}^0 \mathrm{d}\lambda_2 - \int_{-\infty}^0 \mathrm{d}\lambda_1 \int_0^{+\infty} \mathrm{d}\lambda_2\right)[\tfrac{1}{2}\delta(z_{12}^2)(z_1'^\mu z_2'^\nu - z_1'^\nu z_2'^\mu) - \delta'(z_{12}^2)(z_1^\mu z_2^\nu - z_1^\nu z_2^\mu)(z_1'z_2')], \tag{3.2}$$

where $A_b^\mu(x_a^\lambda)$ is the half-retarded plus half-advanced Lienard–Wiechert potential produced by charge b at x_a^λ, given by

$$A_b^\mu(x_a^\lambda) = e_b \int_{-\infty}^{+\infty} \mathrm{d}\lambda_b \delta[(x_a - z_b)^2] z_b'^\mu$$
$$\left(z_a'^\mu \equiv \frac{\mathrm{d}z_a^\mu}{\mathrm{d}\lambda_a}, \qquad \text{while} \qquad \dot z_a^\mu \equiv \frac{\mathrm{d}z_a^\mu}{\mathrm{d}\tau_a}\right). \tag{3.3}$$

We may rewrite formulae (3.1)–(3.3) without using the δ function and its derivative. For it, we define $\hat\lambda_b^\epsilon(\lambda_a)$ ($\epsilon = -, +$) as follows:

$$\{z_a(\lambda_a) - z_b[\hat\lambda_b^\epsilon(\lambda_a)]\}^2 \equiv 0, \qquad \operatorname{sgn}\{z_b^0[\hat\lambda_b^\epsilon(\lambda_a)] - z_a^0(\lambda_a)\} \equiv \epsilon. \tag{3.4}$$

It is not difficult to see that (3.1)–(3.3) are equivalent to

$$p^\mu = \sum_a [m_a u_a^\mu + e_a A_b^\mu(x_a^\lambda)] + \frac{e_1e_2}{2}\sum_\epsilon \left[-\frac{z_{12}^\mu(z_1'z_2')}{|(z_1'z_{12})||(z_2'z_{12})|}\bigg|_{\lambda_1=\hat\lambda_1^\epsilon(0),\,\lambda_2=0} + \int_0^{\hat\lambda_1^\epsilon(0)} \mathrm{d}\lambda_1 |(z_2'z_{12})|^{-1}\frac{\partial}{\partial\lambda_2}\left(\frac{z_{12}^\mu(z_1'z_2')}{(z_2'z_{12})}\right)\bigg|_{\lambda_2=\hat\lambda_2^{-\epsilon}(\lambda_1)}\right] \tag{3.5}$$

$$L^{\mu\nu} = \sum_a \{x_a^\mu[m_a u_a^\nu + e_a A_b^\nu(x_a^\lambda)] - x_a^\nu[m_a u_a^\mu + e_a A_b^\mu(x_a^\lambda)]\} + \frac{e_1e_2}{2}\sum_\epsilon\left[\frac{(z_1^\mu z_2^\nu - z_1^\nu z_2^\mu)(z_1'z_2')}{|(z_1'z_{12})||(z_2'z_{12})|}\bigg|_{\lambda_1=\hat\lambda_1^\epsilon(0),\,\lambda_2=0} + \int_0^{\hat\lambda_1^\epsilon(0)} \mathrm{d}\lambda_1 |(z_2'z_{12})|^{-1} \times \left(z_1'^\mu z_2'^\nu - z_1'^\nu z_2'^\mu - \frac{\partial}{\partial\lambda_2}\frac{(z_1^\mu z_2^\nu - z_1^\nu z_2^\mu)(z_1'z_2')}{(z_2'z_{12})}\right)\bigg|_{\lambda_2=\hat\lambda_2^{-\epsilon}(\lambda_1)}\right] \tag{3.6}$$

$$A_b^\mu(x_a^\lambda) = \frac{1}{2}e_b\sum_\epsilon \frac{z_b'^\mu}{|(z_b'z_{ab})|}\bigg|_{\lambda_a=0,\,\lambda_b=\hat\lambda_b^\epsilon(0)} \tag{3.7}$$

The four-momentum P^μ and the angular momentum $L^{\mu\nu}$ can be written, similarly to (2.5), in the form

$$P^\mu = Dx^\mu_{12} + E^c u^\mu_c + H\eta^{\mu\alpha\beta\gamma} x_{12\alpha} u_{1\beta} u_{2\gamma} \qquad (H = 0 \text{ in PRM}) \tag{3.8}$$

$$L^{\mu\nu} = -D(x^\mu_1 x^\nu_2 - x^\nu_1 x^\mu_2) + E^{cd}(x^\mu_c u^\nu_d - x^\nu_c u^\mu_d) + F(u^\mu_1 u^\nu_2 - u^\nu_1 u^\mu_2), \tag{3.9}$$

with

$$E^{1c} + E^{2c} = E^c. \tag{3.10}$$

The relations between the coefficients of P^μ and some of the coefficients of $L^{\mu\nu}$ can be obtained for instance from the following relation (Schild 1963)

$$L^{\mu\nu}_{(v)} = L^{\mu\nu}_{(0)} - V^\mu P^\nu + V^\nu P^\mu,$$

where $L^{\mu\nu}_{(0)}$ and $L^{\mu\nu}_{(v)}$ are the angular momenta about the origin and about the event V^μ respectively.

4. First-order results

Up to now the formulae are completely exact, and no use has been made of PRM. This section will be devoted to the calculation of P^μ and $L^{\mu\nu}$ to order $g \equiv e_1 e_2$. This approximation corresponds to the substitution of the rectilinear tangent trajectories for the real ones, as may be easily seen.

From (2.18), $z^{(0)\alpha}_b = x^\alpha_b + \lambda_b u^\alpha_b$, which substituted into (3.4) gives

$$\hat{\lambda}^{(0)\epsilon}_b(\lambda_a) = k\lambda_a - y_b + \epsilon\chi_a(\lambda_a), \tag{4.1}$$

with

$$\chi_a(\lambda_a) \equiv [r^2_b + 2s_b\lambda_a + (k^2-1)\lambda^2_a]^{1/2}. \tag{4.2}$$

The four-potentials are, to zero order,

$$A^{(0)\mu}_b(x^\lambda_a) = e_b \frac{u^\mu_b}{r_b}, \tag{4.3}$$

and a straightforward calculation using (3.5) and (3.6) yields the following results for the zero- and first-order coefficients:

$$D^{(0)} = 0, \qquad E^{(0)a} = E^{(0)aa} = m_a, \qquad E^{(0)ab} = 0, \qquad F^{(0)} = 0 \tag{4.4}$$

$$\left.\begin{aligned} D^{(1)} &= \frac{k}{p^2}\left(\frac{s_1}{r_1} - \frac{s_2}{r_2}\right) \\ E^{(1)a} &= E^{(1)ba} = \frac{kr_1r_2 - r^2_2 - s^2_2 - ks_2y_2}{p^2 r_a}, \qquad E^{(1)aa} = 0 \\ F^{(1)} &= \frac{k(s_1r_2 - s_2r_1)}{(k^2-1)p^2} + \frac{1}{(k^2-1)^{3/2}} \ln\frac{(k^2-1)^{1/2}r_2 + s_2}{(k^2-1)^{1/2}r_1 + s_1} \end{aligned}\right\} \tag{4.5}$$

where

$$p^2 \equiv (k^2-1)r^2_1 - s^2_1 = (k^2-1)r^2_2 - s^2_2 \geqslant 0 \tag{4.6}$$

($(k^2-1)^{-1/2}p$ is the minimum interval between the tangents to the trajectories at x^μ_1, x^ν_2).

5. Second-order results

In order to obtain P^{μ} and $L^{\mu\nu}$ up to order g^2, it is first necessary to compute $\hat{\lambda}_b(\lambda_a)$ up to order g. From (2.8) and definition (3.4), a little calculation gives

$$\hat{\lambda}_b^{(1)\epsilon}(\lambda_a) = \epsilon\chi_a^{-1}\{[\rho_a^{(1)}+\rho_b^{(1)}(\hat{\lambda}_b^{(0)\epsilon})](s_b\lambda_a+r_b^2-\epsilon y_b\chi_a)$$
$$+[\sigma_{aa}^{(1)}-\sigma_{ba}^{(1)}(\hat{\lambda}_b^{(0)\epsilon})][(k^2-1)\lambda_a+s_b+\epsilon k\chi_a]+[\sigma_{ab}^{(1)}-\sigma_{bb}^{(1)}(\hat{\lambda}_b^{(0)\epsilon})]\epsilon\chi_a\}. \quad (5.1)$$

We have now all the basic elements needed to obtain $P^{(2)\mu}$ and $L^{(2)\mu\nu}$ from equations (3.5)–(3.7). However, the calculations are extremely involved, and we shall only compute the leading terms in the expansion in power series of the velocities. Time-reversal invariance indicates that the coefficients D and F are homogeneous functions of odd degree and the E's are homogeneous functions of even degree of the velocities. Therefore, we are only concerned with the values of the E's when the velocities are zero ('static case'), and with the terms of D linearly dependent on the velocities. Since F is the coefficient of $u_1^{\mu}u_2^{\nu}-u_1^{\nu}u_2^{\mu}$, and it is consequently always multiplied by at least one velocity, we shall not compute it.

In the rest of this section we assume that the events x_1^{μ} and x_2^{μ} are simultaneous, that is, $x_{12}^0 = 0$. Then we have

$$k = 1+O(v^2), \qquad r_b = x+O(v^2), \qquad s_b = \boldsymbol{x}_{12}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)+O(v^3),$$
$$y_b = \boldsymbol{x}_{ab}\cdot\boldsymbol{v}_{ab}+O(v^3) \quad (5.2)$$

with

$$x \equiv \sqrt{(x_{12}x_{12})} = \sqrt{(\boldsymbol{x}_{12}\cdot\boldsymbol{x}_{12})}.$$

A lengthy and tedious calculation yields:

$$D^{(2)} = \tfrac{1}{4}x^{-4}\sum_a m_a^{-1}(-\boldsymbol{v}_a+5\boldsymbol{v}_b)\cdot\boldsymbol{x}_{12}+O(v^3)$$
$$E^{(2)a} = \tfrac{1}{2}x^{-2}(m_a^{-1}-m_b^{-1})+O(v^2)$$
$$E^{(2)aa} = -\tfrac{3}{4}x^{-2}m_b^{-1}+O(v^2) \quad (5.3)$$
$$E^{(2)ab} = \tfrac{1}{4}x^{-2}(m_a^{-1}+2m_b^{-1})+O(v^2)$$
$$F^{(2)} = O(v).$$

6. Conclusion

For reference purposes we give here the complete results we have obtained, restoring the c's in the formulae:

$$P^{\mu} = m_1u_1^{\mu}+m_2u_2^{\mu}+\frac{g}{p^2}\left[ck\left(\frac{s_1}{r_1}-\frac{s_2}{r_2}\right)x_{12}^{\mu}+(kr_1r_2-r_2^2-s_2^2-ks_2y_2)\left(\frac{u_1^{\mu}}{r_1}+\frac{u_2^{\mu}}{r_2}\right)\right]$$
$$+\frac{g^2}{x^2}\left(\frac{1}{4x^2}\{[m_1^{-1}(-\boldsymbol{v}_1+5\boldsymbol{v}_2)+m_2^{-1}(-\boldsymbol{v}_2+5\boldsymbol{v}_1)]\cdot\boldsymbol{x}_{12}+cO(\beta^3)\}x_{12}^{\mu}\right.$$
$$\left.+\frac{1}{2}(m_1^{-1}-m_2^{-1})(u_1^{\mu}-u_2^{\mu})+O(\beta^2)u_1^{\mu}+O(\beta^2)u_2^{\mu}\right)+O(g^3), \quad (6.1)$$

$$L^{\mu\nu} = m_1(x_1^\mu u_1^\nu - x_1^\nu u_1^\mu) + m_2(x_2^\mu u_2^\nu - x_2^\nu u_2^\mu) + \frac{g}{p^2}\left\{ck\left(\frac{s^2}{r^2} - \frac{s^1}{r^1}\right)(x_1^\mu x_2^\nu - x_1^\nu x_2^\mu)\right.$$
$$+ (kr_1r_2 - r_2^2 - s_2^2 - ks_2y_2)\left(\frac{1}{r_2}(x_1^\mu u_2^\nu - x_1^\nu u_2^\mu) + \frac{1}{r_1}(x_2^\mu u_1^\nu - x_2^\nu u_1^\mu)\right)$$
$$\left. + \frac{1}{c}\left[\frac{k(s_1r_2 - s_2r_1)}{k^2-1} + \frac{p^2}{(k^2-1)^{3/2}}\ln\left(\frac{\sqrt{(k^2-1)}r_2 + s_2}{\sqrt{(k^2-1)}r_1 + s_1}\right)\right](u_1^\mu u_2^\nu - u_1^\nu u_2^\mu)\right\}$$
$$+ \frac{g^2}{x^2}\left(\frac{1}{4x^2}\{[m_1^{-1}(\boldsymbol{v}_1 - 5\boldsymbol{v}_2) + m_2^{-1}(\boldsymbol{v}_2 - 5\boldsymbol{v}_1)] \cdot \boldsymbol{x}_{12} + cO(\beta^3)\}(x_1^\mu x_2^\nu - x_1^\nu x_2^\mu)\right.$$
$$- [\tfrac{3}{4}m_2^{-1} + O(\beta^2)](x_1^\mu u_1^\nu - x_1^\nu u_1^\mu) - [\tfrac{3}{4}m_1^{-1} + O(\beta^2)](x_2^\mu u_2^\nu - x_2^\nu u_2^\mu)$$
$$+ \tfrac{1}{4}[m_1^{-1} + 2m_2^{-1} + O(\beta^2)](x_1^\mu u_2^\nu - x_1^\nu u_2^\mu) + \tfrac{1}{4}[m_2^{-1} + 2m_1^{-1} + O(\beta^2)]$$
$$\left. \times (x_2^\mu u_1^\nu - x_2^\nu u_1^\mu) + \frac{1}{c}O(\beta)(u_1^\mu u_2^\nu - u_1^\nu u_2^\mu)\right) + O(g^3), \qquad (6.2)$$

with

$$g \equiv \frac{e_1e_2}{c^2} \qquad \text{(electrostatic or Gaussian system)}$$

or

$$g \equiv e_1e_2 \qquad \text{(electromagnetic system)},$$
$$k \equiv -\frac{1}{c^2}(u_1u_2) \geqslant 1, \qquad r_b \equiv \left(x^2 + \frac{1}{c^2}(xu_b)^2\right)^{1/2}, \qquad (6.4)$$
$$s_b \equiv \frac{1}{c}(x_{ab}u_a) + \frac{1}{c^3}(u_1u_2)(x_{ab}u_b), \qquad y_b \equiv \frac{1}{c}(x_{ab}u_b),$$
$$p^2 \equiv (k^2-1)r_b^2 - s_b^2,$$
$$(u^\alpha) \equiv (1-\beta^2)^{-1/2}(c, \boldsymbol{v}), \qquad \beta \equiv \frac{v}{c}.$$

The coefficients of g^2 have been calculated for $x_{12}^0 \equiv x_1^0 - x_2^0 = 0$.

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